SAMPLE **SAMPLE**

March 2025 [AUTHOR: Bueler/Rhodes]

Optimization Comprehensive Exam

DO **SIX** OF THE FOLLOWING EIGHT PROBLEMS; CLEARLY INDICATE WHICH ONES.

A. **(a)** Consider the linear programming problem

$$\begin{aligned} \text{minimize} \quad & z = -2x_1 - 3x_2 \\ \text{subject to} \quad & -x_1 + x_2 \ge -3 \\ & -2x_1 + x_2 \le 1 \\ & x_1 + x_2 \le 7 \\ & x_1 \ge 0,\, x_2 \ge 0 \end{aligned}$$

Sketch the feasible set S for the above problem. Be sure to label the axes, and **give the coordinates for each extreme point of** S.

(b) Put the above problem in standard form.

(c) Which extreme point solves the problem?

(d) If you start at the extreme point $(0,0)$, and apply the usual simplex method, which will be the next extreme point to be visited?

B. Let $\mathcal{E}$ and $\mathcal{I}$ be finite sets of indices. Consider a feasible set defined by linear equality and inequality constraints:

$$S = \left\{ x \in \mathbb{R}^n \,:\, \begin{array}{ll} a_i^T x = b_i & \text{for } i \text{ in } \mathcal{E} \\ a_i^T x \ge b_i & \text{for } i \text{ in } \mathcal{I} \end{array} \right\}$$

For each i, we have $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$.

(a) Show that S is convex.

Now assume that $\tilde{x} \in S$ is a feasible point. Note that your conditions in the next two parts will depend on a_i, b_i, and $\tilde{x}$.

(b) By definition, $p \in \mathbb{R}^n$ is a feasible direction at $\tilde{x}$ if there is $\epsilon > 0$ so that $x = \tilde{x} + \alpha p$ is feasible ($x \in S$) for all $0 \le \alpha \le \epsilon$. Identify necessary and sufficient conditions on $p \in \mathbb{R}^n$ so that p is a feasible direction.

(c) Suppose $p \in \mathbb{R}^n$ is a feasible direction at $\tilde{x}$. Identify necessary and sufficient conditions on $\alpha \ge 0$ so that $x = \tilde{x} + \alpha p$ is feasible, that is, $x \in S$. (*Hint. These conditions also depend on* p*.*)

C. Suppose x is a point of a set $S = \{x \in \mathbb{R}^n \,:\, Ax = b, x \ge 0\}$, where A is an $m \times n$ matrix with $m \le n$, and $b \in \mathbb{R}^m$. Show that if x is a basic feasible solution then it is an extreme point of S.

D. Consider the 2D unconstrained minimization problem

$$\text{minimize} \quad f(x) = 3\sin(x_1) + \cos(x_2) + \frac{1}{20}\left(x_1^2 - x_1 x_2 + 2x_2^2\right).$$

(a) Compute the gradient and the Hessian.

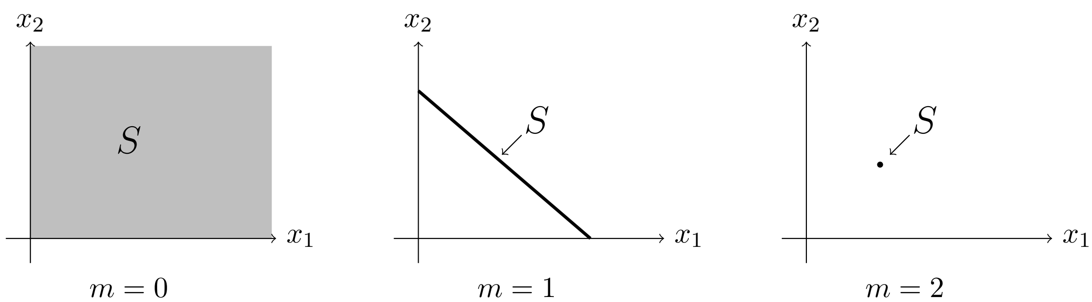
(b) The surface $z = f(x)$ has many local maxima and minima. Choose an algorithm which would, in good circumstances, quickly find a local minimum, and write a pseudocode for this algorithm. Specifically, this algorithm should use the Hessian *and* it should be guaranteed to converge to a critical point.

(c) Describe an algorithm which would find all of the local minima in a closed rectangle, for example $R = \{-100 \le x_1 \le 100, -100 \le x_2 \le 100\}$.

E. Assume $f : \mathbb{R}^n \to \mathbb{R}$ is smooth, $b \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m\times n}$ with $m \le n$. Consider nonlinear optimization problems which have standard-form linear constraints:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b \\ & x \ge 0 \end{array}$$

In 2D ($n = 2$) there are three possibilities for the dimension of the feasible set. The cartoons below illustrate these three possibilities in the cases where the feasible sets S are non-empty, generic, and bounded when $m > 0$.



For 3D ($n = 3$) there are four possibilities $m = 0, 1, 2, 3$. Sketch the four corresponding cartoons. These cartoons should have the same annotations as the 2D versions above.

F. Consider the problem

$$\begin{array}{ll} \text{minimize} & f(x) = (x_1 - 1)^2 + (x_2 + 1)^2 \\ \text{subject to} & x_1^2 + x_2^2 \le 4 \\ & x_2 \ge 0 \end{array}$$

(a) Sketch the feasible set and explain informally, perhaps using contours of f, why $x_* = (1, 0)^\top$ is the solution.

(b) Write the constraints in the form $g_i(x) \ge 0$. Compute the Lagrangian and its gradient.

(c) For each of the points $A = (0, 0)^\top$, $B = (0, 2)^\top$, and $C = (1, 0)^\top$, compute the values of the Lagrange multipliers λ_i satisfying the zero-gradient condition. Do any of these points satisfy the first-order optimality conditions?

G. Suppose $c \in \mathbb{R}^n$ is a nonzero vector and consider the problem

$$\begin{array}{ll} \text{minimize} & z = c^\top x \\ \text{subject to} & \displaystyle\sum_{i=1}^n x_i^2 = 1 \end{array}$$

where $x \in \mathbb{R}^n$. Note that the equality constraint can also be written $\|x\|^2 = 1$.

(a) Compute the Lagrangian, and its gradient, and state the first-order necessary conditions.

(b) Solve the first-order conditions algebraically. How many points (x_*, λ_*) satisfy the first-order necessary conditions? What is the solution to the problem?

H. Consider the nonlinear constrained optimization problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) = 0, \quad i = 1, \dots, \ell \\ & h_i(x) \geq 0, \quad i = 1, \dots, m \end{array}$$

(a) Assume f, g_i, h_i are all smooth. State the Lagrangian for this problem.

(b) Suppose x_* is a local minimizer. State the complete first-order necessary conditions for the above problem. That is, give the general KKT conditions.