Real Analysis Comprehensive Exam

Complete **EIGHT** of the following ten problems. It is better to fully complete fewer problems than to earn partial credit on many problems.

1. Compute, with justification,
\[
\lim_{n \to \infty} \int_0^1 \sqrt{\frac{nx}{1 + n^2 x^2}} \, dx.
\]

2. Give examples of the following
   (a) A sequence \((f_n)\) in \(C[0,1]\) that converges pointwise to 0 but such that \(\int_0^1 f_n \neq 0\).
   (b) A bounded sequence in \(\ell_1\) that has no convergent subsequence.
   (c) A sequence in \(C[0,1]\) that converges pointwise to a discontinuous function.

3. Let \((f_n)\) be a bounded sequence of functions in \(C[0,1]\). Define
\[
F_n(x) = \int_0^x f_n(s) \, ds.
\]
Show that \((F_n)\) has a uniformly convergent subsequence.

4. (a) State the Axiom of Completeness for \(\mathbb{R}\).
   (b) Prove that a monotone increasing sequence of real numbers converges if and only if it is bounded above.

5. Let \(m^*\) denote Lebesgue outer measure. Suppose that \(E\) is a subset of \(\mathbb{R}\) such that \(m^*(E \cap (a,b)) \leq 3/4 (b-a)\) for every interval \((a,b)\). Prove that \(m^*(E) = 0\).

6. Let \((a_n)\) be a bounded sequence. Show that \(\sum_{n=1}^{\infty} a_n e^{-nx}\) defines a continuous function on \([1,2]\).

7. Suppose that \(X\) is compact and \(f : X \to \mathbb{R}\) is continuous. Prove that \(f\) is uniformly continuous.

8. Suppose \(f \in L^1(\mathbb{R})\) is uniformly continuous. Show that \(\lim_{x \to \infty} f(x) = 0\).

9. Let \(f \in L^1[-\pi, \pi]\). Show that
\[
\lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx = 0.
\]
   Hint: First consider the case where \(f\) is the characteristic function of an interval.

10. Let \(f : [0,1] \to \mathbb{R}\) be an increasing, but not necessarily continuous function. Show that \(f\) is Borel measurable.