Numerical Linear Algebra Comprehensive Exam

**PART I: DO ALL OF THE PROBLEMS A–F**

A. Suppose $A \in \mathbb{C}^{m \times m}$ is invertible. Consider the factorization $PA = LU$ for $L$ unit lower-triangular, $U$ upper-triangular, and $P$ a permutation matrix.

(a) Given $b \in \mathbb{C}^m$, how is this factorization used to solve $Ax = b$ for $x \in \mathbb{C}^m$?

(b) Give leading order estimates, as $m \to \infty$, of the number of floating point operations to implement the major stages of the method in part (a), including the cost of the factorization $PA = LU$. Assume the factorization is computed by Gaussian elimination with partial pivoting and that the other major steps use standard algorithms. (*Name these standard algorithms.*)

(c) Is Gaussian elimination with partial pivoting always the best method for solving a square system like $Ax = b$? Describe at least one alternative algorithm with superior stability properties.

B. (a) Suppose $A \in \mathbb{C}^{m \times n}$. Define a *singular value decomposition* (SVD) of $A$. For $m > n$, describe how the *reduced SVD* differs from the SVD.

(b) For $m > n$ and $A$ of full rank, use the reduced SVD to construct an orthogonal projector $P$ onto the range of $A$. Show that $P$ is an orthogonal projector.

C. (a) Compute $\|A\|_2$ and $\|A\|_F$ for the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

(b) Compute the condition number $\text{cond}(A) = \kappa(A)$ using the 2-norm.

D. Let $\|\cdot\|$ be any norm on $\mathbb{C}^m$ and also let it denote the corresponding induced norm on square matrices $A \in \mathbb{C}^{m \times m}$. Show that $\rho(A) \leq \|A\|$ if $\rho(A)$ is the spectral radius of $A$. (*The spectral radius of $A$ is the maximum of $|\lambda|$ over the eigenvalues $\lambda$ of $A$.*)

E. Consider a computer satisfying the standard axioms for floating point arithmetic,\(^1\) so that the machine precision $\epsilon_{\text{mach}}$ is precisely defined. Let $X, Y$ be real, normed vector spaces and let $f : X \to Y$ be a problem. Precisely define: The algorithm $\hat{f} : X \to Y$ computing the problem $f$ is *backward stable*. (An informal definition might be included to explain the idea, but it does not suffice.)

F. Give an example of an invertible $2 \times 2$ matrix $A$ which has $\det(A) > 10^{20}$ but for which the system $Ax = b$ is well-conditioned.

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\(^1\)Namely that for each such computer there exists $\epsilon_{\text{mach}} > 0$ so that the following two facts hold: (1) For all $x \in \mathbb{R}$ there is $\epsilon$ so that $|\epsilon| \leq \epsilon_{\text{mach}}$ and $fl(x) = x(1 + \epsilon)$. (2) For all $x, y \in \mathbb{R}$ and each operation $* = +, -, \times, \div$, with computer implementation $\odot$, there is $\epsilon$ so that $|\epsilon| \leq \epsilon_{\text{mach}}$ and $x \odot y = (x * y)(1 + \epsilon)$. 
Part II: do TWO of the following three problems

1. (a) Let
   \[ A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}. \]
   Compute the eigenvalues and eigenvectors of \( A \).

   (b) Let \( x \in \mathbb{C}^2 \) be a random vector. For the same matrix \( A \), estimate
   \[ \frac{\| A^{2013} x \|}{\| A^{2012} x \|}. \]
   Explain. Include in your explanation a description of those rare vectors for which your estimate is not accurate.

2. Suppose we define a square matrix \( A \in \mathbb{C}^{m \times m} \) to be normal if there is an orthonormal basis of \( \mathbb{C}^m \) consisting of eigenvectors of \( A \).
   (a) Show that if \( A \) is normal then \( A^* A = AA^* \).
   (b) Show that if \( A \) is normal then any Schur factorization of \( A \) is, in fact, a unitary diagonalization.

3. (a) Show that if \( P \) is an orthogonal projector then \( I - 2P \) is unitary.
   (b) For \( A \in \mathbb{C}^{m \times n} \) of full rank with \( m \geq n \), \( A^* \) the hermitian transpose of \( A \), \( b \in \mathbb{C}^m \), and \( P \) the orthogonal projector onto the range of \( A \), show that the equations
   \[ A^* Ax = A^* b \quad \text{ and } \quad Ax = Pb \]
   have the same unique solution \( x \in \mathbb{C}^n \).